4. **Proposed by H. W. Holycross, Superintendent of Schools, Pottersburg, Union County, Ohio.**

What value of x will render $4x^4+12x^3-3x^2-2x+1$ a square?

Solution by P. S. BERG, Apple Creek, Ohio.

Extracting the square root of the expression we get $2x^2+3x-3$ as the partial root and a remainder of $16x-8$, or the expression $=(2x^2+3x-3)^2+16x-8$. Now when $16x-8=0$, the expression is a square. $\therefore$ $x=\frac{1}{2}$ is the required value.

Also solved by *J. H. Drummond*, *A. L. Foote*, *H. C. Whitaker*, *G. B. M. Zerr*, and *O. S. Kibler*.

PROBLEMS.

5. **Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.**

Find three numbers the sum of the squares of any two of which diminished by their product shall be a square number.

6. **Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Find three whole numbers the sum of any two of which is a cube.

8. **Proposed by Hon. JOSIAH H. DRUMMOND, Portland, Maine.**

Every odd square is of the form $4a+1$; find the value of a for the nth consecutive odd square.

Solutions to these problems should be received on or before May 1st.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

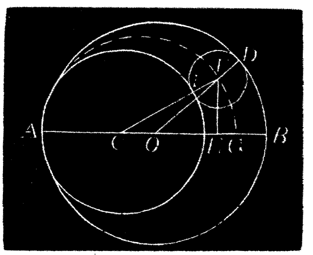
SOLUTION TO A PROBLEM IN AVERAGE.

By B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Two circles whose radii are R and r respectively, are tangent internally. Find the average area of all circles that can be drawn tangent to the two circles.

Let $AO=R$ be the radius of the larger circle; $AC=r$, the radius of the smaller circle; and A the point of internal tangency of the two circles.

Let ID be any circle inscribed within the crescent, F the center of this circle, $IF=z$, its radius, and (x, y) the rectangular co-ordinates of the point F referred to A as the origin of co-ordinates. Draw FC and OD. Then $OF=(R-z)$, $CF=(r+z)$, $CE=(x-r)$, and $OE=(x-R)$. From the figure, we have $FC^2-CE^2=OF^2-OE^2$, or $(r+z)^2-(x-r)^2=(R-z)^2-(x-R)^2$, whence $z=\left(\dfrac{R-r}{R+r}\right)x$. We also have $OF^2=OE^2+EF^2$, or $(R-z)^2=(x-R)^2+y^2$. Substituting the value of z and solving with respect to y^2, we have $y^2=\dfrac{4Rr}{(R+r)^2}[(R+r)x-x^2]$,



which is the rectangular equation of an ellipse referred to its left hand vertex as origin. Changing this to an equation referred to the center of the ellipse and substituting a for $\frac{1}{2}(R+r)$ and b for $\sqrt{Rr}$, we have $y^2+(1-e^2)x^2=a^2(1-e^2)$.

$z=\left(\frac{R-r}{R+r}\right)[\frac{1}{2}(R+r)+x]=e(a+x)$, referred to the center of the ellipse.

The centers of the inscribed circles are uniformly distributed on the circumference of this ellipse, and the number of circles is, therefore, proportional to the circumference of the ellipse.

Let s = the length of any portion of the circumference of the ellipse, measured from the extremity of the transverse diameter.

Then the *average area* sought is

$$\Delta=\frac{\int \pi z^2 ds}{\int ds}. \quad \text{But } ds=\frac{dx}{(a^2-x^2)^{\frac{1}{2}}}\times(a^2-e^2x^2)^{\frac{1}{2}} \text{ and } z=e(a+x),$$

$$\text{where } e=\left(\frac{R-r}{R+r}\right). \quad \therefore\ \Delta=\frac{\pi e^2}{aE(e)}\int_{-a}^{a}\frac{(a+x)^2dx}{\sqrt{(a^2-x^2)}}\times(a^2-e^2x^2)^{\frac{1}{2}}$$

$$=\frac{\pi e^2}{aE(e)}\int_{-a}^{a}\left[\frac{a^2(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}+\frac{2ax(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}+\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}\right]dx,$$

$$=\frac{\pi e^2}{aE(e)}\left[a^3E(e)+\int_{-a}^{a}\frac{2ax(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}+\int_{-a}^{a}\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}\right].$$

The value of the second term [in the above equation] is zero between the limits $x=-a$ and $x=a$. If we let $x=av$, we have for the third term,

$$\int_{-a}^{a}\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}=a^3\int_{-1}^{1}\frac{v^2(1-e^2v^2)^{\frac{1}{2}}dv}{\sqrt{(1-v^2)}}$$

$$=a^3\left[\int_{-2}^{1}\left\{\frac{(1-e^2v^2)^{\frac{1}{2}}}{\sqrt{(1-v^2)}}-(1-v^2)^{\frac{1}{2}}(1-e^2v^2)^{\frac{1}{2}}\right\}dv\right]$$

$$=a^3\left[E(e)-\frac{[1-(1+e^2)v^2+e^2v^4]dv}{\sqrt{(1-v^2)}\sqrt{(1-e^2v^2)}}\right]=a^3\left[E(e)-\tfrac{1}{3}[v(1-v^2)(1-e^2v^2)^{\frac{1}{2}}\right]_{-1}^{1}$$

$$+\tfrac{1}{3}\int_{-1}^{1}\frac{[2-(1+e^2)v^2]dv}{\sqrt{(1-v)}\sqrt{(1-e^2v^2)}}\Bigg]=a^3\left[E(e)+\frac{1}{3e^2}(1+e^2)\int_{-1}^{1}\frac{(1-e^2v^2)^{\frac{1}{2}}dv}{\sqrt{(1-v^2)}}\right.$$

$$\left.-\frac{1}{3e^2}\int_{-1}^{1}\frac{dv}{\sqrt{(1-v^2)}\sqrt{(1-e^2v^2)}}\right]=a^3\left[E(e)-\frac{2}{3e^2}(1+e^2)E(e)+\tfrac{2}{3}(1-e^2)F(e)\right].$$

$$\therefore\ \Delta=\frac{\pi e^2}{aE(e)}\left\{a^3E(e)+a^3\left[E(e)-\frac{2}{3e^2}(1+e^2)E(e)+\tfrac{2}{3}(1-e^2)F(e)\right]\right\},$$

$$=\frac{\pi}{3}a^2\left[4e^2-2+(1-e^2)\left(\frac{F(e)}{E(e)}\right)\right]=\frac{\pi}{3}\left[(R-r)^2-Rr+Rr\left(\frac{F(e)}{E(e)}\right)\right].$$